IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Lie bialgebra quantizations of the oscillator algebra and their universal ***R***-matrices

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1996 J. Phys. A: Math. Gen. 29 4307

(http://iopscience.iop.org/0305-4470/29/15/006)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.70
The article was downloaded on 02/06/2010 at 03:56

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **29** (1996) 4307–4320. Printed in the UK

Lie bialgebra quantizations of the oscillator algebra and their universal R-matrices

Angel Ballesteros and Francisco J Herranz
Departamento de Física, Universidad de Burgos, Pza Misael Bañuelos, E-09001, Burgos, Spain

Received 25 January 1996

Abstract. All coboundary Lie bialgebras and their corresponding Poisson–Lie structures are constructed for the oscillator algebra generated by $\{N, A_+, A_-, M\}$. Quantum oscillator algebras are derived from these bialgebras by using the Lyakhovsky and Mudrov formalism and, for some cases, quantizations at both algebra and group levels are obtained, including their universal R-matrices.

1. Introduction

Deformed Heisenberg and oscillator algebras have recently been the focus of many investigations coming from various directions. Among them, we would like to quote the construction of deformed statistics [1], the use of q-Heisenberg algebras to describe composite particles [2], the description of certain classes of exactly solvable potentials in terms of a q-Heisenberg dynamical symmetry [3], the link between deformed oscillator algebras and superintegrable systems [4, 5] and the relations between these deformed algebras and q-orthogonal polynomials [6].

Quantum universal enveloping algebras (QUEAs) are much more selective deformations than general modifications of the commutation rules of a given algebra. In particular, the interest of finding Hopf algebra deformations of the oscillator algebra is twofold: first, because of the relevant role played by Hopf algebras to build up second quantization, as has been recently discussed in [7], and second, a quasitriangular quantum oscillator algebra has been related to Yang–Baxter systems and link invariants in [8].

The aim of this paper is to provide a systematic study of the quantum universal enveloping oscillator algebras underlying possible further generalizations of these results. A brief summary of the oscillator algebra and group is given in section 2. Since every QUEA defines uniquely a Lie bialgebra structure on the undeformed algebra, in section 3 we obtain and classify all coboundary Lie bialgebra structures for the harmonic oscillator algebra, as well as their corresponding Poisson–Lie brackets. In section 4 we make use of the Lyakhovsky and Mudrov formalism [9] in order to build up the deformed coproducts linked to all these Lie harmonic oscillator bialgebras. A complete quantization (including universal R-matrices) of two particular classes of non-standard (triangular) bialgebras is provided: the former is the natural 'extension' of the non-standard deformation of the $1+1$ Poincaré algebra discussed in [10] and the latter is a new three parameter quantization.

To our knowledge, the literature on Hopf algebra deformations of the oscillator algebra includes only the deformation given in [8, 11] and some new results that have been recently given in [12] by computing the dual of an arbitrary quantum oscillator group obtained by

 © 1996 IOP Publishing Ltd

following an R-matrix approach in a particular matrix representation (see [13–15]). Among these known deformations, the former can be easily included within our classification at the Lie bialgebra level, and can thus be obtained without making use of contraction procedures. On the other hand, our method gives explicit (and universal) expressions for the oscillator QUE algebras linked to the quantizations of [12] which are coboundaries. The procedure outlined here precludes cumbersome duality computations and leads to rather simple candidates for universal R-matrices.

2. Classical oscillator algebra and group

The oscillator Lie algebra h_4 is generated by $\{N, A_+, A_-, M\}$ with Lie brackets

$$[N, A_+] = A_+ \qquad [N, A_-] = -A_- \qquad [A_-, A_+] = M \qquad [M, \cdot\,] = 0. \tag{2.1}$$

Besides the central generator M there exists another Casimir invariant

$$C = 2NM - A_+A_- - A_-A_+. \tag{2.2}$$

A 3×3 real matrix representation $\mathbf{D}$ of (2.1) is given by:

$$\mathbf{D}(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{D}(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

The expression for a generic element of the oscillator group H_4 coming from this representation is:

$$\begin{aligned} \mathbf{T}^D &= \exp\{m\mathbf{D}(M)\}\exp\{a_-\mathbf{D}(A_-)\}\exp\{a_+\mathbf{D}(A_+)\}\exp\{n\mathbf{D}(N)\} \\ &= \begin{pmatrix} 1 & a_-\mathrm{e}^N & m + a_-a_+ \\ 0 & \mathrm{e}^N & a_+ \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.4}$$

The group law for the coordinates m, a_-, a_+ and n is obtained by means of matrix multiplication $\mathbf{T}^{D''} = \mathbf{T}^{D'} \cdot \mathbf{T}^D$:

$$\begin{aligned} n'' &= n + n', \qquad & m'' &= m + m' - a_-a'_+\,\mathrm{e}^{-n'} \\ a''_+ &= a'_+ + a_+\,\mathrm{e}^{n'} \qquad & a''_- &= a'_- + a_-\,\mathrm{e}^{-n'}. \end{aligned} \tag{2.5}$$

Left and right invariant vector fields are also deduced from (2.4) and read

$$X^L_N = \partial_N \qquad X^L_{A_+} = \mathrm{e}^N\partial_{a_+} \qquad X^L_{A_-} = \mathrm{e}^{-N}\partial_{a_-} - a_+\,\mathrm{e}^{-N}\partial_m \qquad X^L_M = \partial_m \tag{2.6}$$

$$X^R_N = \partial_N + a_+\partial_{a_+} + a_-\partial_{a_-} \qquad X^R_{A_+} = \partial_{a_+} - a_-\partial_m \qquad X^R_{A_-} = \partial_{a_-} \qquad X^R_M = \partial_m. \tag{2.7}$$

The Heisenberg algebra can be seen as the subalgebra $\langle A_+, A_-, M\rangle$ of h_4 and the Heisenberg group $\langle a_+, a_-, m\rangle$ is recovered by taking the coordinate $n \equiv 0$ in H_4. Moreover, h_4 can be seen as a centrally extended (1+1) Poincaré algebra (by M). This fact will be useful in the quantization process.

3. Coboundary oscillator Lie bialgebras

Let g be a Lie algebra and let r be an element of $g \wedge g$. The cocomutator $\delta : g \to g \wedge g$ given by

$$\delta(X) = [1 \otimes X + X \otimes 1,\ r] \qquad X \in g \tag{3.1}$$

defines a coboundary Lie bialgebra $(g, \delta(r))$ if and only if r fulfills the modified classical Yang–Baxter equation (YBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]\,] = 0 \qquad X \in g \tag{3.2}$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \tag{3.3}$$

and, if $r = r^{ij} X_i \otimes X_j$, we have denoted $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$.

When the r-matrix is such that $[[r, r]] = 0$ (classical YBE), we shall say that $(g, \delta(r))$ is a *non-standard* (or triangular) Lie bialgebra. In contrast, a solution r of (3.2) with non-vanishing Schouten bracket will give rise to a so-called *standard* Lie bialgebra.

We recall that, if $g = \mathrm{Lie}(G)$, the (unique) Poisson–Lie structure on $C^\infty(G)$ linked to a fixed bialgebra $(g, \delta(r))$ is given by the Sklyanin bracket

$$\{\Psi, \Phi\} = r^{\alpha\beta} \left(X^L_\alpha \Psi X^L_\beta \Phi - X^R_\alpha \Psi X^R_\beta \Phi \right) \qquad \Psi, \Phi \in C^\infty(G) \tag{3.4}$$

where X^L_α and X^R_τ are the left and right invariant vector fields of G, respectively.

In particular, for h_4 we shall consider an arbitrary element r, which can be written in terms of six (real) coefficients:

$$r = \alpha_+ N \wedge A_+ + \alpha_- N \wedge A_- + \vartheta\, N \wedge M + \xi\, A_+ \wedge A_- + \beta_+ A_+ \wedge M + \beta_- A_- \wedge M. \tag{3.5}$$

It is a matter of computation to prove that the corresponding Schouten bracket for r (3.5) is

$$\begin{aligned}[][[r, r]] = {} & \alpha_+ (\xi + \vartheta)\, N \wedge M \wedge A_+ + \alpha_- (\xi - \vartheta)\, N \wedge M \wedge A_- \\ & -2\alpha_+\alpha_- N \wedge A_+ \wedge A_- + (\alpha_+\beta_- + \alpha_-\beta_+ - \xi^2)\, M \wedge A_+ \wedge A_-. \end{aligned} \tag{3.6}$$

From this expression it follows that the modified classical YBE (3.2) is fulfilled provided that

$$\alpha_+\alpha_- = 0 \qquad \alpha_+(\xi + \vartheta) = 0 \qquad \alpha_-(\xi - \vartheta) = 0. \tag{3.7}$$

The solutions of this system are splitted into three classes: $\alpha_+ \neq 0$, $\alpha_- \neq 0$ and $\alpha_+ = \alpha_- = 0$. For each of them we shall distinguish between non-standard ($[[r, r]] = 0$) and standard Lie bialgebras as follows.

Type I_+. If $\alpha_+ \neq 0$ we have $\alpha_- = 0$ and $\xi = -\vartheta$. The Schouten bracket reduces to:

$$[[r, r]] = (\alpha_+\beta_- - \vartheta^2)\, (M \wedge A_+ \wedge A_-). \tag{3.8}$$

Therefore if $\alpha_+\beta_- - \vartheta^2 \neq 0$ we have standard solutions and when $\beta_- = \vartheta^2/\alpha_+$ we are considering non-standard ones.

Type I_-. If $\alpha_- \neq 0$ equations (3.7) imply $\alpha_+ = 0$ and $\xi = \vartheta$. The Schouten bracket is now

$$[[r, r]] = (\alpha_-\beta_+ - \vartheta^2)(M \wedge A_+ \wedge A_-). \tag{3.9}$$

Standard solutions are obtained when $\alpha_-\beta_+ - \vartheta^2 \neq 0$, while non-standard ones correspond to $\beta_+ = \vartheta^2/\alpha_-$.

Table 1. Coboundary oscillator Lie bialgebras.

	Type I_+	
	Standard ($\alpha_+ \neq 0$ and $\alpha_+\beta_- - \vartheta^2 \neq 0$)	Non-standard ($\alpha_+ \neq 0$)
r	$\alpha_+ N\wedge A_+ + \vartheta(N\wedge M - A_+\wedge A_-)$ $+\beta_+ A_+\wedge M + \beta_- A_-\wedge M$	$\alpha_+ N\wedge A_+ + \vartheta(N\wedge M - A_+\wedge A_-)$ $+\beta_+ A_+\wedge M + (\vartheta^2/\alpha_+)A_-\wedge M$
$\delta(N)$	$\alpha_+ N\wedge A_+ - \beta_- A_-\wedge M + \beta_+ A_+\wedge M$	$\alpha_+ N\wedge A_+ - (\vartheta^2/\alpha_+)A_-\wedge M + \beta_+ A_+\wedge M$
$\delta(A_+)$	0	0
$\delta(A_-)$	$\alpha_+(A_-\wedge A_+ + N\wedge M) + 2\vartheta A_-\wedge M$	$\alpha_+(A_-\wedge A_+ + N\wedge M) + 2\vartheta A_-\wedge M$
$\delta(M)$	0	0
	Type I_-	
	Standard ($\alpha_- \neq 0$ and $\alpha_-\beta_+ - \vartheta^2 \neq 0$)	Non-standard ($\alpha_- \neq 0$)
r	$\alpha_- N\wedge A_- + \vartheta(N\wedge M + A_+\wedge A_-)$ $+\beta_+ A_+\wedge M + \beta_- A_-\wedge M$	$\alpha_- N\wedge A_- + \vartheta(N\wedge M + A_+\wedge A_-)$ $+(\vartheta^2/\alpha_-)A_+\wedge M + \beta_- A_-\wedge M$
$\delta(N)$	$-\alpha_- N\wedge A_- + \beta_+ A_+\wedge M - \beta_- A_-\wedge M$	$-\alpha_- N\wedge A_- + (\vartheta^2/\alpha_-)A_+\wedge M - \beta_- A_-\wedge M$
$\delta(A_+)$	$-\alpha_-(A_+\wedge A_- + N\wedge M) - 2\vartheta A_+\wedge M$	$-\alpha_-(A_+\wedge A_- + N\wedge M) - 2\vartheta A_+\wedge M$
$\delta(A_-)$	0	0
$\delta(M)$	0	0
	Type II	
	Standard ($\xi \neq 0$)	Non-standard
r	$\vartheta N\wedge M + \xi A_+\wedge A_-$ $+\beta_+ A_+\wedge M + \beta_- A_-\wedge M$	$\vartheta N\wedge M + \beta_+ A_+\wedge M + \beta_- A_-\wedge M$
$\delta(N)$	$\beta_+ A_+\wedge M - \beta_- A_-\wedge M$	$\beta_+ A_+\wedge M - \beta_- A_-\wedge M$
$\delta(A_+)$	$-(\vartheta+\xi)A_+\wedge M$	$-\vartheta A_+\wedge M$
$\delta(A_-)$	$(\vartheta-\xi)A_-\wedge M$	$\vartheta A_-\wedge M$
$\delta(M)$	0	0

Type II. Finally, we consider the case with $\alpha_+ = 0$; if $\alpha_- \neq 0$ we are again in type I_-, so we must also take $\alpha_- = 0$ in order to have three disjoint sets of solutions. In this case equations (3.7) are automatically satisfied and the Schouten bracket is

$$[[r, r]] = -\xi^2\, M\wedge A_+\wedge A_-. \tag{3.10}$$

Then the condition $\xi \neq 0$ gives rise to standard solutions and $\xi = 0$ to non-standard ones.

All the information concerning this classification of coboundary oscillator Lie bialgebras is summarized in table 1. Poisson–Lie structures for the oscillator group are deduced via the Sklyanin bracket (3.4) and presented in table 2.

Note that this classification is based in the use of skew-symmetric r-matrices. This implies no loss of generality: given an arbitrary element of $g\otimes g$, the map δ generated by (3.1) has to be skew-symmetric to give rise to a Lie bialgebra. This amounts to impose $Ad^{\otimes 2}$-invariance on the symmetric part of r and, therefore, r will generate the same Lie bialgebra as its skew-symmetric part [16]. In particular, it can be easily checked that the more general element η of $h_4\otimes h_4$ such that

$$[X\otimes 1 + 1\otimes X, \eta] = 0 \qquad X \in \{N, A_+, A_-, M\} \tag{3.11}$$

is given by

$$\eta = \tau_1(N\otimes M + M\otimes N - A_+\otimes A_- - A_-\otimes A_+) + \tau_2\, M\otimes M \tag{3.12}$$

i.e. a linear combination of two terms directly related to the two Casimirs of h_4.

Table 2. Poisson–Lie brackets on the oscillator group.

	Type I_+ Standard ($\alpha_+ \neq 0$ and $\alpha_+\beta_- - \vartheta^2 \neq 0$)	Type I_+ Non-standard ($\alpha_+ \neq 0$)
$\{n, a_+\}$	$\alpha_+(e^n - 1)$	$\alpha_+(e^n - 1)$
$\{n, a_-\}$	0	0
$\{a_-, a_+\}$	$\alpha_+ a_-$	$\alpha_+ a_-$
$\{n, m\}$	$\alpha_+ a_-$	$\alpha_+ a_-$
$\{a_+, m\}$	$\alpha_+ a_- a_+ + \beta_+(e^n - 1)$	$\alpha_+ a_- a_+ + \beta_+(e^n - 1)$
$\{a_-, m\}$	$-\alpha_+ a_-^2 + 2\vartheta a_- + \beta_-(e^{-n} - 1)$	$-\alpha_+ a_-^2 + 2\vartheta a_- + (\vartheta^2/\alpha_+)(e^{-n} - 1)$
	Type I_- Standard ($\alpha_- \neq 0$ and $\alpha_-\beta_+ - \vartheta^2 \neq 0$)	Type I_- Non-standard ($\alpha_- \neq 0$)
$\{n, a_+\}$	0	0
$\{n, a_-\}$	$\alpha_-(e^{-n} - 1)$	$\alpha_-(e^{-n} - 1)$
$\{a_-, a_+\}$	$\alpha_- a_+$	$\alpha_- a_+$
$\{n, m\}$	$-\alpha_- a_+ e^{-n}$	$-\alpha_- a_+ e^{-n}$
$\{a_+, m\}$	$-2\vartheta a_+ + \beta_+(e^N - 1)$	$-2\vartheta a_+ + (\vartheta^2/\alpha_-)(e^n - 1)$
$\{a_-, m\}$	$\beta_-(e^{-n} - 1)$	$\beta_-(e^{-n} - 1)$
	Type II Standard ($\xi \neq 0$)	Type II Non-standard
$\{n, a_+\}$	0	0
$\{n, a_-\}$	0	0
$\{a_-, a_+\}$	0	0
$\{n, m\}$	0	0
$\{a_+, m\}$	$-(\vartheta + \xi)a_+ + \beta_+(e^n - 1)$	$-\vartheta a_+ + \beta_+(e^n - 1)$
$\{a_-, m\}$	$(\vartheta - \xi)a_- + \beta_-(e^{-n} - 1)$	$\vartheta a_- + \beta_-(e^{-n} - 1)$

4. Quantization

In this section we first show how the Lyakhovsky and Mudrov (LM) formalism [9] allows all the cocommutators of the oscillator bialgebras previously found to generate coassociative coproducts in a straightforward way. Afterwards, we shall construct commutation rules and universal quantum R-matrices for some of these bialgebra quantizations.

4.1. The Lyakhovsky–Mudrov formalism

Let us start with a short resume of the LM formalism which applies to an associative algebra E over $\mathbb{C}$ with unit and generated by n commuting elements H_i and m additional elements X_j. For any $m \times m$ numerical matrix $\boldsymbol{\mu}$, by $\boldsymbol{\mu} H$ we understand the matrix $\boldsymbol{\mu}$ with all its entries multiplied by H. If P is an $m \times m$ matrix with entries $p_{kl} \in E$, the kth component of $\mathsf{P} \dot{\otimes} \boldsymbol{X}$ is defined as

$$(\mathsf{P} \dot{\otimes} \boldsymbol{X})_k = \sum_{l=1}^{m} p_{kl} \otimes X_l. \tag{4.1}$$

The main LM statement [9] is that E can be endowed with a coalgebra structure as follows (where we have denoted by σ the permutation map $\sigma(a \otimes b) = b \otimes a$).

Proposition 1. Let $\{1, H_1, \ldots, H_n, X_1, \ldots, X_m\}$ a basis of an associative algebra E over $\mathbb{C}$ verifying the conditions

$$[H_i, H_j] = 0 \qquad i, j = 1, \ldots, n. \tag{4.2}$$

Let μ_i, ν_j $(i, j = 1, \ldots, n)$ be a set of $m \times m$ complex matrices such that

$$[\boldsymbol{\mu}_i, \boldsymbol{\nu}_j] = [\boldsymbol{\mu}_i, \boldsymbol{\mu}_j] = [\boldsymbol{\nu}_i, \boldsymbol{\nu}_j] = 0 \qquad i, j = 1, \ldots, n. \tag{4.3}$$

Let $\boldsymbol{X}$ be a column vector with components X_l $(l = 1, \ldots, m)$. The coproduct and the counit

$$\Delta(1) = 1 \otimes 1 \qquad \Delta(H_i) = 1 \otimes H_i + H_i \otimes 1$$

$$\Delta(\boldsymbol{X}) = \exp\left(\sum_{i=1}^{n} \boldsymbol{\mu}_i H_i\right) \dot{\otimes} \boldsymbol{X} + \sigma\left(\exp\left(\sum_{i=1}^{n} \boldsymbol{\nu}_i H_i\right) \dot{\otimes} \boldsymbol{X}\right) \tag{4.4}$$

$$\epsilon(1) = 1 \qquad \epsilon(H_i) = \epsilon(X_l) = 0 \qquad i = 1, \ldots, n;\ l = 1, \ldots m \tag{4.5}$$

endow (E, Δ, ϵ) with a coalgebra structure.

The resulting coalgebra can be seen as a multiparametric deformation where the deformation parameters are the entries of the matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$. If we are able to find a compatible multiplication with the coproduct (4.4) we will have finally obtained a quantum algebra.

It is worth remarking that this formalism encodes in the set of matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$ the whole coalgebra structure. In fact, the role of these matrices is, essentially, to reflect the Lie bialgebra underlying a given quantum deformation. This can be clearly appreciated by taking the first order (in all the parameters) of (4.4):

$$\Delta_{(1)}(\boldsymbol{X}) = \left(\sum_{i=1}^{n} \boldsymbol{\mu}_i H_i\right) \dot{\otimes} \boldsymbol{X} + \sigma\left(\left(\sum_{i=1}^{n} \boldsymbol{\nu}_i H_i\right) \dot{\otimes} \boldsymbol{X}\right) \tag{4.6}$$

and recalling that the cocommutator δ corresponds to the co-antisymmetric part of (4.6). It can be written in 'matrix' form as

$$\delta(\boldsymbol{X}) = \Delta_{(1)}(\boldsymbol{X}) - \sigma \circ \Delta_{(1)}(\boldsymbol{X}). \tag{4.7}$$

We would like to emphasize the following points.

- The commuting elements H_i are the primitive generators.
- The cocommutator $\delta(X_i)$ does not contain terms of the form $H_i \wedge H_j$.
- The same cocommutator (4.7) can be obtained from different choices of the matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$. This means that different sets of matrices might lead to right quantizations, all of them having the same first-order terms in the deformation parameters. Moreover, we can choose $\boldsymbol{\mu}_i = 0$ as a representative of all these quantizations and we shall obtain

$$\delta(\boldsymbol{X}) = -\left(\sum_{i=1}^{n} \boldsymbol{\nu}_i H_i\right) \dot{\wedge} \boldsymbol{X} = -\left(\sum_{i=1}^{n} \boldsymbol{\nu}_i H_i\right) \dot{\otimes} \boldsymbol{X} + \sigma\left(\sum_{i=1}^{n} \boldsymbol{\nu}_i H_i\right) \dot{\otimes} \boldsymbol{X}. \tag{4.8}$$

Now let us reverse somehow the LM formalism to try to find in which way the oscillator Lie bialgebras given in table 1 can be recovered by a suitable choice of the matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$. Of course, the benefit of such a situation is to be able to 'exponentiate' directly the bialgebra (4.8) to a full coalgebra (4.4)–(4.5).

Let us start with non-standard type I_+ oscillator bialgebras. By denoting $H_1 \equiv A_+$, $H_2 \equiv M$, $X_1 \equiv N$, $X_2 \equiv A_-$, we see that $[H_1, H_2] = 0$ and there exists a term of the type $H_1 \wedge H_2 \equiv A_+ \wedge M$ within the cocommutator $\delta(N)$; however, this obstruction can be circumvented by defining a new generator in the form

$$N' = N - (\beta_+/\alpha_+)M. \tag{4.9}$$

Hence, the cocommutators for the non-primitive generators N' and A_- can be written as

$$\delta\begin{pmatrix} N' \\ A_- \end{pmatrix} = \begin{pmatrix} -\alpha_+ A_+ & 0 \\ 0 & -\alpha_+ A_+ \end{pmatrix} \dot{\wedge} \begin{pmatrix} N' \\ A_- \end{pmatrix} + \begin{pmatrix} 0 & (\vartheta^2/\alpha_+)M \\ -\alpha_+ M & -2\vartheta M \end{pmatrix} \dot{\wedge} \begin{pmatrix} N' \\ A_- \end{pmatrix}. \tag{4.10}$$

In view of this expression, the matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$ can be chosen as

$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{\nu}_1 = \begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_+ \end{pmatrix} \qquad \boldsymbol{\nu}_2 = \begin{pmatrix} 0 & -\vartheta^2/\alpha_+ \\ \alpha_+ & 2\vartheta \end{pmatrix}. \tag{4.11}$$

Now, the set of conditions of proposition 1 are fulfilled, and we can use this result to get the coproducts:

$$\begin{aligned} \Delta\begin{pmatrix} N' \\ A_- \end{pmatrix} &= \exp\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix} N' \\ A_- \end{pmatrix} \\ &\quad +\sigma\left(\exp\left\{\begin{pmatrix} \alpha_+ A_+ & -(\vartheta^2/\alpha_+)M \\ \alpha_+ M & \alpha_+ A_+ + 2\vartheta M \end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix} N' \\ A_- \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 \otimes N' + N' \otimes (1 - \vartheta M)\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} - (\vartheta^2/\alpha_+) A_- \otimes M\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} \\ 1 \otimes A_- + A_- \otimes (1 + \vartheta M)\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} + \alpha_+ N' \otimes M\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} \end{pmatrix}. \end{aligned} \tag{4.12}$$

We can finally return to the initial basis elements, thus obtaining a three-parameter QUEA (denoted by $U^{(I_+ n)}_{\alpha_+,\vartheta,\beta_+}(h_4)$) such that

$$\begin{aligned} \Delta(N) &= 1 \otimes N + N \otimes (1 - \vartheta M)\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} - (\vartheta^2/\alpha_+) A_- \otimes M\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} \\ &\quad + (\beta_+/\alpha_+) M \otimes (1 - (1 - \vartheta M)\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M}) \\ \Delta(A_-) &= 1 \otimes A_- + A_- \otimes (1 + \vartheta M)\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M} \\ &\quad + (\alpha_+ N - \beta_+ M) \otimes M\,\mathrm{e}^{\alpha_+ A_+ + \vartheta M}. \end{aligned} \tag{4.13}$$

This quantization procedure can be applied to the remaining types of bialgebras in the same way. For the standard type I_+ bialgebras we also use (4.9), while for the bialgebras of type I_- we introduce the new generator

$$N' = N - (\beta_-/\alpha_-)M. \tag{4.14}$$

In contrast, no such a kind of transformation is necessary to get the coproducts for the Lie bialgebras of type II.

The coproducts for the corresponding QUEA of the coboundary oscillator Lie bialgebras of table 1 are written down in table 3; we denote each multiparametric quantum coalgebra by $U^{(t\,m)}_{\alpha_i}(h_4)$ where t is the type, $m = s$ or $m = n$ according either to the standard or non-standard oscillator deformations with α_i being the deformation parameters. The explicit expressions for the coproducts of $U^{(\mathrm{I}_+ s)}_{\alpha_+,\vartheta,\beta_+,\beta_-}(h_4)$ and $U^{(\mathrm{I}_- s)}_{\alpha_-,\vartheta,\beta_+,\beta_-}(h_4)$ are rather complicated so we keep their matrix forms written in terms of the generator N' defined by either (4.9) or by (4.14), respectively.

The final step in the quantization process of a fixed bialgebra is to find the commutation relations compatible with its deformed coproduct (the counit and antipode can be obtained in the form explained in [9]). In the following, we solve completely this problem and construct the deformed Hopf algebras $U^{(t\,m)}_{\alpha_i}(h_4)$ for some representative cases among the ones included in table 3.

Table 3. Coproducts for QUEA of the oscillator algebra

Type I_+

Standard: $U^{(\mathrm{I}_+s)}_{\alpha_+,\vartheta,\beta_+,\beta_-}(h_4)$ $\quad(\alpha_+ \neq 0$ and $\alpha_+\beta_- - \vartheta^2 \neq 0)$

$$\Delta(A_+) = 1\otimes A_+ + A_+\otimes 1 \quad \Delta(M) = 1\otimes M + M\otimes 1$$

$$\Delta\begin{pmatrix} N' \\ A_- \end{pmatrix} = \begin{pmatrix} 1\otimes N' \\ 1\otimes A_- \end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix} \alpha_+A_+ & -\beta_-M \\ \alpha_+M & \alpha_+A_+ + 2\vartheta M \end{pmatrix}\right\}\dot{\otimes}\begin{pmatrix} N' \\ A_- \end{pmatrix}\right)$$

$$N' = N - (\beta_+/\alpha_+)M$$

Non-standard: $U^{(\mathrm{I}_+n)}_{\alpha_+,\vartheta,\beta_+}(h_4)$ $\quad(\alpha_+ \neq 0)$

$$\Delta(A_+) = 1\otimes A_+ + A_+\otimes 1\,\Delta(M) = 1\otimes M + M\otimes 1$$

$$\Delta(N) = 1\otimes N + N\otimes(1-\vartheta M)\exp\{\alpha_+A_+ + \vartheta M\} - (\vartheta^2/\alpha_+)A_-\otimes M\exp\{\alpha_+A_+ + \vartheta M\}$$
$$+(\beta_+/\alpha_+)M\otimes(1-(1-\vartheta M)\exp\{\alpha_+A_+ + \vartheta M\})$$

$$\Delta(A_-) = 1\otimes A_- + A_-\otimes(1+\vartheta M)\exp\{\alpha_+A_+ + \vartheta M\} + (\alpha_+N - \beta_+M)\otimes M\exp\{\alpha_+A_+ + \vartheta M\}$$

Type I_-

Standard: $U^{(\mathrm{I}_-s)}_{\alpha_-,\vartheta,\beta_+,\beta_-}(h_4)$ $\quad(\alpha_- \neq 0$ and $\alpha_-\beta_+ - \vartheta^2 \neq 0)$

$$\Delta(A_-) = 1\otimes A_- + A_-\otimes 1\,\Delta(M) = 1\otimes M + M\otimes 1$$

$$\Delta\begin{pmatrix} N' \\ A_+ \end{pmatrix} = \begin{pmatrix} 1\otimes N' \\ 1\otimes A_+ \end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix} -\alpha_-A_- & \beta_+M \\ -\alpha_-M & -\alpha_-A_- - 2\vartheta M \end{pmatrix}\right\}\dot{\otimes}\begin{pmatrix} N' \\ A_+ \end{pmatrix}\right)$$

$$N' = N - (\beta_-/\alpha_-)M$$

Non-standard: $U^{(\mathrm{I}_-n)}_{\alpha_-,\vartheta,\beta_-}(h_4)$ $\quad(\alpha_- \neq 0)$

$$\Delta(A_-) = 1\otimes A_- + A_-\otimes 1\,\Delta(M) = 1\otimes M + M\otimes 1$$

$$\Delta(N) = 1\otimes N + N\otimes(1+\vartheta M)\exp\{-\alpha_-A_- - \vartheta M\} + (\vartheta^2/\alpha_-)A_+\otimes M\exp\{-\alpha_-A_- - \vartheta M\}$$
$$+(\beta_-/\alpha_-)M\otimes(1-(1+\vartheta M)\exp\{-\alpha_-A_- - \vartheta M\})$$

$$\Delta(A_+) = 1\otimes A_+ + A_+\otimes(1-\vartheta M)\exp\{-\alpha_-A_- - \vartheta M\} - (\alpha_-N - \beta_+M)\otimes M\exp\{-\alpha_-A_- - \vartheta M\}$$

Type II

Standard: $U^{(\mathrm{II}s)}_{\vartheta,\xi,\beta_+,\beta_-}(h_4)$ $\quad(\xi \neq 0)$

$$\Delta(M) = 1\otimes M + M\otimes 1$$
$$\Delta(A_+) = 1\otimes A_+ + A_+\otimes\exp\{-(\vartheta+\xi)M\}$$
$$\Delta(A_-) = 1\otimes A_- + A_-\otimes\exp\{(\vartheta-\xi)M\}$$
$$\Delta(N) = 1\otimes N + N\otimes 1 + (\beta_+/(\vartheta+\xi))A_+\otimes(1-\exp\{-(\vartheta+\xi)M\}) + (\beta_-/(\vartheta-\xi))A_-$$
$$\otimes(1-\exp\{(\vartheta-\xi)M\})$$

Non-standard: $U^{(\mathrm{II}n)}_{\vartheta,\beta_+,\beta_-}(h_4)$

$$\Delta(M) = 1\otimes M + M\otimes 1$$
$$\Delta(A_+) = 1\otimes A_+ + A_+\otimes\exp\{-\vartheta M\}$$
$$\Delta(A_-) = 1\otimes A_- + A_-\otimes\exp\{\vartheta M\}$$
$$\Delta(N) = 1\otimes N + N\otimes 1 + (\beta_+/\vartheta)A_+\otimes(1-\exp\{-\vartheta M\}) + (\beta_-/\vartheta)A_-\otimes(1-\exp\{\vartheta M\})$$

4.2. Non-standard type I_+: $U_z^{(n)}(h_4)$

It is remarkable that the oscillator algebra with basis $\{N, A_+, A_-, M\}$ can be interpreted as an extended (1+1) Poincaré algebra where N is the boost generator, A_+ and A_- generate the translations along the light-cone and M is the central generator. This fact raises the question about whether it is possible to implement in this extended case the universal (non-standard) quantum deformation of the Poincaré algebra studied in [10] from a T-matrix approach.

Let us consider the non-standard oscillator bialgebras of type I_+ with $\vartheta = \beta_+ = 0$ and

$\alpha_+ \equiv z$. According to table 1 the Lie bialgebra is characterized by commutation relations (2.1), the classical r-matrix

$$r = z\,N \wedge A_+ \tag{4.15}$$

and cocommutators:

$$\delta(A_+) = 0 \qquad \delta(M) = 0 \qquad \delta(N) = zN \wedge A_+$$
$$\delta(A_-) = z(A_- \wedge A_+ + N \wedge M). \tag{4.16}$$

Poisson–Lie brackets are easily deduced from table 2:

$$\{n, a_+\} = z(e^n - 1) \qquad \{n, a_-\} = 0 \qquad \{a_-, a_+\} = za_-$$
$$\{n, m\} = za_- \qquad \{a_+, m\} = za_-a_+ \qquad \{a_-, m\} = -za_-^2. \tag{4.17}$$

A quantum deformation for this Lie bialgebra is given by the following statement.

Proposition 2. The coproduct Δ, counit ϵ, antipode γ

$$\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1 \qquad \Delta(M) = 1 \otimes M + M \otimes 1$$
$$\Delta(N) = 1 \otimes N + N \otimes \mathrm{e}^{zA_+} \qquad \Delta(A_-) = 1 \otimes A_- + A_- \otimes \mathrm{e}^{zA_+} + zN \otimes M\mathrm{e}^{zA_+} \tag{4.18}$$

$$\epsilon(X) = 0 \qquad X \in \{N, A_+, A_-, M\} \tag{4.19}$$

$$\gamma(A_+) = -A_+ \qquad \gamma(M) = -M$$
$$\gamma(N) = -N\,\mathrm{e}^{-zA_+} \qquad \gamma(A_-) = -A_-\,\mathrm{e}^{-zA_+} + zNM\mathrm{e}^{-zA_+} \tag{4.20}$$

and the commutation relations

$$[N, A_+] = \frac{\mathrm{e}^{zA_+} - 1}{z} \qquad [N, A_-] = -A_- \qquad [A_-, A_+] = M\mathrm{e}^{zA_+}$$
$$[M, \cdot\,] = 0 \tag{4.21}$$

determine a Hopf algebra (denoted by $U_z^{(n)}(h_4)$) which quantizes the non-standard bialgebra generated by the classical r-matrix (4.15).

The coproduct (4.18) is obtained from table 3. Note that M remains as a central generator. There is another element belonging to the centre of $U_z^{(n)}h_4$ whose classical limit is (2.2), namely

$$C_z = 2NM + \frac{\mathrm{e}^{-zA_+} - 1}{z}A_- + A_-\frac{\mathrm{e}^{-zA_+} - 1}{z}. \tag{4.22}$$

An important feature of the quantum algebra $U_z^{(n)}(h_4)$ is that the generators N and A_+ form a Hopf subalgebra which coincides exactly with the corresponding quantum Poincaré algebra of [10]. We recall that for this Hopf subalgebra there is a universal R-matrix given by

$$R = \exp\{-zA_+ \otimes N\}\exp\{zN \otimes A_+\}. \tag{4.23}$$

Obviously, (4.23) satisfies the quantum YBE for $U_z^{(n)}h_4$, but, moreover, it verifies

$$\sigma \circ \Delta(X) = R\Delta(X)R^{-1} \qquad \text{for } X \in \{N, A_+, A_-, M\}. \tag{4.24}$$

This assertion must be proved only for M and A_-; the proof for the former is trivial since it is a central generator, and for the latter we have

$$\exp\{zN \otimes A_+\}\Delta(A_-)\exp\{-zN \otimes A_+\} = 1 \otimes A_- + A_- \otimes 1$$
$$= \Delta_0(A_-)\exp\{-zA_+ \otimes N\}\Delta_0(A_-)\exp\{zA_+ \otimes N\} = \sigma \circ \Delta(A_-). \tag{4.25}$$

The fulfillment of relation (4.24) allows one to use the FRT approach in order to get a quantum deformation of $\mathrm{Fun}(H_4)$ by taking into account that in the matrix representation (2.3) the universal R-matrix (4.23) collapses into

$$\mathbf{D}(R) = \mathbf{I} \otimes \mathbf{I} + z(\mathbf{D}(N) \otimes \mathbf{D}(A_+) - \mathbf{D}(A_+) \otimes \mathbf{D}(N)) \tag{4.26}$$

where $\mathbf{I}$ is the 3×3 identity matrix. Therefore, the Hopf structure of the associated oscillator quantum group is given by

Proposition 3. The coproduct, counit, antipode

$$\begin{aligned}
&\Delta(\hat{n}) = 1 \otimes \hat{n} + \hat{n} \otimes 1 \\
&\Delta(\hat{a}_+) = \mathrm{e}^{\hat{n}} \otimes \hat{a}_+ + \hat{a}_+ \otimes 1 \\
&\Delta(\hat{a}_-) = \mathrm{e}^{-\hat{n}} \otimes \hat{a}_- + \hat{a}_- \otimes 1 \\
&\Delta(\hat{m}) = 1 \otimes \hat{m} + \hat{m} \otimes 1 - \mathrm{e}^{-\hat{n}}\hat{a}_+ \otimes \hat{a}_-
\end{aligned} \tag{4.27}$$

$$\epsilon(X) = 0 \qquad X \in \{\hat{n}, \hat{a}_+, \hat{a}_-, \hat{m}\} \tag{4.28}$$

$$\begin{aligned}
&\gamma(\hat{n}) = -\hat{n} \qquad \gamma(\hat{a}_+) = -\mathrm{e}^{-\hat{n}}\hat{a}_+ \\
&\gamma(\hat{a}_-) = -\mathrm{e}^{\hat{n}}\hat{a}_- \qquad \gamma(\hat{m}) = -\hat{m} - (\mathrm{e}^{-\hat{n}}\hat{a}_+ \mathrm{e}^{\hat{n}})\hat{a}_-
\end{aligned} \tag{4.29}$$

together with the commutation relations

$$\begin{aligned}
&[\hat{n}, \hat{a}_+] = z\,(\mathrm{e}^{\hat{n}} - 1) \qquad [\hat{n}, \hat{a}_-] = 0 \qquad [\hat{a}_-, \hat{a}_+] = z\hat{a}_- \\
&[\hat{n}, \hat{m}] = z\hat{a}_- \qquad [\hat{a}_+, \hat{m}] = z\hat{a}_-\hat{a}_+ \qquad [\hat{a}_-, \hat{m}] = -z\hat{a}_-^2
\end{aligned} \tag{4.30}$$

constitute a Hopf algebra denoted by $\mathrm{Fun}_z^{(n)}(H_4)$.

The coproduct (4.27), counit (4.28) and antipode (4.29) are obtained from the relations $\Delta(\mathbf{T}) = \mathbf{T} \dot{\otimes} \mathbf{T}$, $\epsilon(\mathbf{T}) = I$ and $\gamma(\mathbf{T}) = \mathbf{T}^{-1}$, where $\mathbf{T} \equiv \mathbf{T}^D$ is the generic element of the oscillator group H_4 (2.4). The commutation rules are deduced from $R\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1 R$, where $\mathbf{T}_1 = \mathbf{T} \otimes \mathbf{I}$, $\mathbf{T}_2 = \mathbf{I} \otimes \mathbf{T}$ and R given by (4.26).

The commutation relations (4.30) can be seen as a Weyl quantization $\{,\} \to z^{-1}[,]$ of the fundamental Poisson brackets (4.17). It is also clear that the coalgebra structure of $\mathrm{Fun}_z^{(n)}(H_4)$ determined by the coproduct (4.27) and counit (4.28) is valid for any quantum group which deforms $\mathrm{Fun}(H_4)$.

Some features of this new quantum oscillator algebra can be emphasized.

• When the central extension M and its corresponding quantum coordinate $\hat{m}$ vanish all results concerning the quantum Poincaré algebra and group given in [10] are recovered. In this sense, the quantum coordinates $\hat{n}$, $\hat{a}_+$ and $\hat{a}_-$ close a quantum Hopf subalgebra which coincides exactly with the quantum Poincaré group just mentioned.

• The primitive generator involved in the deformation is now A_+. This fact will be relevant at a representation theory level and, consequently, from the point of view of the physical properties of this deformed oscillator.

• The deformed Heisenberg subalgebra generated by A_+, A_- and M is not a Hopf subalgebra due to the appearence of N in $\Delta(A_-)$. However, the Hopf subalgebra structure can be recovered by working on a representation where the central generator M is expressed as a multiple of the identity. In this situation, N can be defined in terms of A_+ and A_- by using the Casimir (4.22). In general, this type of non-standard deformed bosons can be expected to build up q-boson realizations of the already known non-standard quantum algebras [17, 18].

4.3. Non-standard type II: $U^{(II\,n)}_{\vartheta,\beta_+,\beta_-}(h_4)$

The classical r-matrix

$$r = \vartheta N \wedge M + \beta_+ A_+ \wedge M + \beta_- A_- \wedge M \tag{4.31}$$

originates a non-standard three-parametric oscillator bialgebra of type II whose cocommutators and associated Poisson–Lie brackets appear in tables 1 and 2, respectively. A quantum deformation of this coboundary Lie bialgebra is given by

Proposition 4. The Hopf algebra denoted by $U^{(\mathrm{II}\,n)}_{\vartheta,\beta_+,\beta_-}(h_4)$ which quantizes the oscillator bialgebra generated by (4.31) has coproduct given in table 3, counit (4.19), antipode

$$\begin{aligned} &\gamma(M) = -M \qquad \gamma(A_+) = -A_+\,\mathrm{e}^{\vartheta M} \qquad \gamma(A_-) = -A_-\,\mathrm{e}^{-\vartheta M} \\ &\gamma(N) = -N - (\beta_+/\vartheta)A_+(1-\mathrm{e}^{\vartheta M}) - (\beta_-/\vartheta)\,A_-(1-\mathrm{e}^{-\vartheta M}) \end{aligned} \tag{4.32}$$

and commutation relations

$$\begin{aligned} &[N, A_+] = A_+ - \beta_- V(-\vartheta) \qquad [N, A_-] = -A_- - \beta_+ V(\vartheta) \\ &[A_-, A_+] = M \qquad [M, \cdot\,] = 0 \end{aligned} \tag{4.33}$$

where

$$V(x) := \frac{1}{x^2}(\mathrm{e}^{xM} - 1 - xM). \tag{4.34}$$

Note that $\lim_{x\to 0} V(x) = M^2/2$. The quantum analogue of (2.2),

$$C_{\vartheta,\beta_+,\beta_-} = 2NM - A_+A_- - A_-A_+ + 2\beta_-\,V(-\vartheta)A_- - 2\beta_+\,V(\vartheta)A_+ \tag{4.35}$$

belongs to the centre of $U^{(\mathrm{II}n)}_{\vartheta,\beta_+,\beta_-}(h_4)$.

It is worth remarking that this quantum oscillator algebra can be related the results of [12]: $U^{(\mathrm{II}\,n)}_{\vartheta,\beta_+,\beta_-}(h_4)$ can be seen as a type II case with $p \equiv \vartheta$, $q \equiv -\vartheta$, $b \equiv \beta_-$ and $c \equiv -\beta_+$. Moreover,

Proposition 5. The element

$$\begin{aligned} R = \exp\{r\} &= \exp\{\vartheta N \wedge M + \beta_+A_+ \wedge M + \beta_-A_- \wedge M\} \\ &= \exp\{-M \otimes (\vartheta N + \beta_+A_+ + \beta_-A_-)\} \\ &\quad \times \exp\{(\vartheta N + \beta_+A_+ + \beta_-A_-) \otimes M\} \end{aligned} \tag{4.36}$$

satisfies both the quantum YBE and relation (4.24), so it is a universal R-matrix for $U^{(\mathrm{II}\,n)}_{\vartheta,\beta_+,\beta_-}(h_4)$.

Since M is a central generator, it is clear that (4.36) is a solution of the quantum YBE. The proof for property (4.24) is sketched in appendix A. In the matrix representation (2.3) we get

$$\mathbf{D}(R) = \mathbf{I}\otimes\mathbf{I} + \vartheta\mathbf{D}(N)\wedge\mathbf{D}(M) + \beta_+\mathbf{D}(A_+)\wedge\mathbf{D}(M) + \beta_-\mathbf{D}(A_-)\wedge\mathbf{D}(M). \tag{4.37}$$

The FRT prescription leads now to another multiparametric quantum deformation of the algebra of the smooth functions on the oscillator group $\mathrm{Fun}^{(\mathrm{II}\,n)}_{\vartheta,\beta_+,\beta_-}(H_4)$, given by coproduct (4.27), counit (4.28), antipode (4.29) and the non-vanishing commutation rules

$$[\hat{a}_+, \hat{m}] = -\vartheta\hat{a}_+ + \beta_+(\mathrm{e}^{\hat{n}} - 1) \qquad [\hat{a}_-, \hat{m}] = \vartheta\hat{a}_- + \beta_-(\mathrm{e}^{-\hat{n}} - 1). \tag{4.38}$$

The classical limit (in the three parameters) is $\mathrm{Fun}(H_4)$ and, once more, commutators (4.38) are a Weyl quantization of the Poisson–Lie brackets written in table 2.

4.4. Standard type II: $U_z^{(s)}(h_4)$

The classical r-matrix which solves the classical YBE and underlies the quantum oscillator algebra obtained in [8, 11] by a contraction method can be expressed in our notation as

$$r = -z\,(N \otimes M + M \otimes N) + 2z\,A_- \otimes A_+. \tag{4.39}$$

Its symmetric (r_+) and skew-symmetric (r_-) parts are

$$r_+ = (r + \sigma \circ r)/2 = z(A_- \otimes A_+ + A_+ \otimes A_-) - z(N \otimes M + M \otimes N) \tag{4.40}$$

$$r_- = (r - \sigma \circ r)/2 = zA_- \wedge A_+. \tag{4.41}$$

The symmetric part r_+ corresponds to the element η (3.12) with the parameters $\tau_1 = -z$ and $\tau_2 = 0$. On the other hand, r_- can be identified with a standard classical r-matrix of type 2 with parameters $\vartheta = \beta_+ = \beta_- = 0$ and $\xi \equiv -z$ (see table 1). Both the standard r-matrix (which coincides with r_- (4.41)) and the non-antisymmetric one (4.39) give rise to the same oscillator bialgebra with cocommutators

$$\delta(N) = \delta(M) = 0 \qquad \delta(A_+) = z\,A_+ \wedge M \qquad \delta(A_-) = z\,A_- \wedge M. \tag{4.42}$$

The associated non-vanishing Poisson–Lie brackets (see table 2) are

$$\{a_+, m\} = z\,a_+ \qquad \{a_-, m\} = z\,a_-. \tag{4.43}$$

The quantum deformation of this coboundary oscillator bialgebra is given by:

Proposition 6. The quantum algebra which quantizes the standard bialgebra generated by (4.39) has a Hopf structure denoted by $U_z^{(s)}(h_4)$ and characterized by the coproduct, counit, antipode

$$\begin{aligned} &\Delta(N) = 1 \otimes N + N \otimes 1 \qquad \Delta(A'_+) = \mathrm{e}^{-zM} \otimes A'_+ + A'_+ \otimes 1 \\ &\Delta(M) = 1 \otimes M + M \otimes 1 \qquad \Delta(A_-) = 1 \otimes A_- + A_- \otimes \mathrm{e}^{zM} \end{aligned} \tag{4.44}$$

$$\epsilon(X) = 0 \qquad X \in \{N, A'_+, A_-, M\} \tag{4.45}$$

$$\gamma(N) = -N \qquad \gamma(M) = -M \qquad \gamma(A'_+) = -A'_+\,\mathrm{e}^{zM} \qquad \gamma(A_-) = -A_-\,\mathrm{e}^{-zM} \tag{4.46}$$

together with the commutation relations

$$[N, A'_+] = A'_+ \qquad [N, A_-] = -A_- \qquad [A_-, A'_+] = \frac{\sinh(zM)}{z} \qquad [M, \cdot\,] = 0. \tag{4.47}$$

The quantum Casimir is

$$C_z = 2N\,\frac{\sinh(zM)}{z} - A'_+A_- - A_-A'_+. \tag{4.48}$$

The coproducts (4.44) are just those given in table 3 but written in terms of a new generator $A'_+ = \mathrm{e}^{\xi M}A_+$ where $\xi = -z$. In this case the universal R-matrix adopts a much simpler form than the one already known from [8, 11]. Namely,

$$\begin{aligned} R &= \exp\{-z(N \otimes M + M \otimes N)\}\exp\{2zA_- \otimes A'_+\} \\ &= \exp\{-zN \otimes M\}\exp\{-zM \otimes N\}\exp\{2z\,A_- \otimes A'_+\}. \end{aligned} \tag{4.49}$$

It is worth remarking that all the quantum R-matrices given in this section are obtained via a straightforward exponentiation process from their classical counterparts (compare, for instance, (4.49) to (4.39)).

The FRT prescription can be applied leading to the commutation rules of the quantum group $\mathrm{Fun}_z^{(s)}(H_4)$ by taking into account that (4.49) in the matrix representation (2.3) is just

$$\mathbf{D}(R) = \mathbf{I} \otimes \mathbf{I} + 2z\mathbf{D}(A_-) \otimes \mathbf{D}(A'_+) - z(\mathbf{D}(N) \otimes \mathbf{D}(M) + \mathbf{D}(M) \otimes \mathbf{D}(N)) \tag{4.50}$$

(note that $\mathbf{D}(A'_+) \equiv \mathbf{D}(N)$). In this way, the non-vanishing commutators of $\mathrm{Fun}_z^{(s)}(H_4)$ read

$$[\hat{a}_+, \hat{m}] = z\hat{a}_+ \qquad [\hat{a}_-, \hat{m}] = z\hat{a}_- \tag{4.51}$$

and correspond to a Weyl quantization of the Poisson–Lie brackets (4.43).

5. Concluding remarks

We have presented a systematic procedure in order to study the coboundary Lie bialgebras of the oscillator algebra. The first-order deformations given by the corresponding cocomutators have been used to construct, by a sort of 'exponentiation' process, multiparametric quantum deformations of the oscillator algebra. We point out that we have not treated the question of the equivalence of the coboundary oscillator bialgebras we have obtained, indeed this is actually a problem by itself. For instance, from an algebraic point of view, bialgebras of types I_+ and I_- can be related by interchanging generators A_+ and A_-, although this result is not so straightforward if we look at their corresponding Poisson–Lie groups.

It is worth stressing that, in the case analysed here, the complete (and rich) classification of the classical r-matrices (and, therefore, of the corresponding Poisson structures on the oscillator group) is easily obtained. This seems to indicate that, at least for Lie algebras with a low enough dimension, the complete solution of the modified classical YBE for an arbitrary skew element of $g \otimes g$ can be explicitly deduced giving rise to a great number of new results.

This kind of procedure is complementary (and dual) to that developed in [12], since it allows us to focus on the deformation at the quantum algebra level and look for universal quantum R-matrices. In fact, given a skew solution r of the modified classical YBE and a matrix representation $\mathbf{D}$ of the quantum algebra, the element $\mathbf{D}(R) = 1 + z\mathbf{D}(r)$ will lead us to the corresponding R-matrix method.

This approach can be seen as a part of research program that, in order to construct and study quantum algebras, tries to extract as much information as possible from the associated Lie bialgebras (as far as contraction methods are concerned, see for instance [19]). It would be interesting to apply it to other physically interesting algebras whose coboundary bialgebra structures are not well known, among them, we would like to mention the Schrödinger, optical and Galilean algebras, also with the aim of obtaining some (universal) quantum deformations.

Acknowledgments

The authors acknowledge M Santander and M A del Olmo for a careful reading of the manuscript and many helpful suggestions as well as the referees for some pertinent comments. This work has been partially supported by DGICYT (Projects PB92–0255 and PB94–1115) from the Ministerio de Educación y Ciencia de España.

Appendix

The main steps necessary to prove that the R-matrix (4.36) verifies the property (4.24) for the generators A_+, A_- and N (for M the proof is trivial) are as follows. We perform the computations by writing the R-matrix in terms of two exponentials $R = \exp\{-M \otimes W\}\exp\{W \otimes M\}$, where $W \equiv \vartheta N + \beta_+ A_+ + \beta_- A_-$. We note that

$$
\begin{aligned}
&\exp\{W\otimes M\}\Delta(A_+)\exp\{-W\otimes M\}\\
&\qquad = 1\otimes A_+ + A_+\otimes 1 - \beta_- V(-\vartheta)\otimes(1-\mathrm{e}^{-\vartheta M}) + (\beta_-/\vartheta)M\otimes(1-\mathrm{e}^{-\vartheta M})\\
&\qquad = \Delta_0(A_+) + (\beta_-/\vartheta^2)(1-\mathrm{e}^{-\vartheta M})\otimes(1-\mathrm{e}^{-\vartheta M}).
\end{aligned}
\tag{A.1}
$$

Since the second term of (A.1) is central, we compute

$$
\begin{aligned}
&\exp\{-M\otimes W\}\Delta_0(A_+)\exp\{M\otimes W\}\\
&\qquad = \mathrm{e}^{-\vartheta M}\otimes A_+ + A_+\otimes 1 + \beta_-(1-\mathrm{e}^{-\vartheta M})\\
&\qquad\quad \otimes V(-\vartheta) - (\beta_-/\vartheta)(1-\mathrm{e}^{-\vartheta M})\otimes M\\
&\qquad = \sigma\circ\Delta(A_+) - (\beta_-/\vartheta^2)(1-\mathrm{e}^{-\vartheta M})\otimes(1-\mathrm{e}^{-\vartheta M}).
\end{aligned}
\tag{A.2}
$$

From these expressions $R\Delta(A_+)R^{-1} = \sigma\circ\Delta(A_+)$ is easily derived. The proof for A_- is rather similar, and for the generator N we shall have

$$
\begin{aligned}
&\exp\{W\otimes M\}\Delta(N)\exp\{-W\otimes M\}\\
&\qquad = 1\otimes N + N\otimes 1 - (\beta_+\beta_-/\vartheta^2)\left\{\vartheta V(\vartheta)\otimes(1-\mathrm{e}^{\vartheta M}) + M\otimes(1-\mathrm{e}^{-\vartheta M})\right\}\\
&\qquad\quad +(\beta_+\beta_-/\vartheta^2)\{\vartheta V(-\vartheta)\otimes(1-\mathrm{e}^{\vartheta M}) - M\otimes(1-\mathrm{e}^{-\vartheta M})\}\\
&\qquad = \Delta_0(N) + (\beta_+\beta_-/\vartheta^3)\{(1-\mathrm{e}^{\vartheta M})\\
&\qquad\quad \otimes(1-\mathrm{e}^{\vartheta M}) - (1-\mathrm{e}^{-\vartheta M})\otimes(1-\mathrm{e}^{-\vartheta M})\}.
\end{aligned}
\tag{A.3}
$$

Now we compute

$$
\begin{aligned}
&\exp\{-M\otimes W\}\Delta_0(N)\exp\{M\otimes W\}\\
&\qquad = \Delta_0(N) + (\beta_+/\vartheta)(1-\mathrm{e}^{-\vartheta M})\otimes A_+ + (\beta_-/\vartheta)(1-\mathrm{e}^{\vartheta M})\otimes A_-\\
&\qquad\quad +(\beta_+\beta_-/\vartheta)\{(1-\mathrm{e}^{\vartheta M})\otimes V(\vartheta) - (1-\mathrm{e}^{-\vartheta M})\otimes V(-\vartheta)\}\\
&\qquad\quad +(\beta_+\beta_-/\vartheta^2)(2-\mathrm{e}^{\vartheta M}-\mathrm{e}^{-\vartheta M})\otimes M\\
&\qquad = \sigma\circ\Delta(N) - (\beta_+\beta_-/\vartheta^3)\{(1-\mathrm{e}^{\vartheta M})\\
&\qquad\quad \otimes(1-\mathrm{e}^{\vartheta M}) - (1-\mathrm{e}^{-\vartheta M})\otimes(1-\mathrm{e}^{-\vartheta M})\}
\end{aligned}
\tag{A.4}
$$

to obtain again $R\Delta(N)R^{-1} = \sigma\circ\Delta(N)$.

References

[1] Greenberg O W 1991 *Phys. Rev.* D **43** 4111
[2] Avancini S S and Krein G 1995 *J. Phys. A: Math. Gen.* **28** 685
[3] Spiridonov V 1992 *Phys. Rev. Lett.* **69** 398
[4] Bonatsos D, Daskaloyannis and Kokkotas K 1993 *Phys. Rev.* A **48** R3407
[5] Bonatsos D, Daskaloyannis and Kokkotas K 1994 *Phys. Rev.* A **50** 3700
[6] Floreanini R, Le Tourneaux J and Vinet L 1995 *J. Phys. A: Math. Gen.* **28** L287
[7] Celeghini E, Rasetti M and Vitiello G 1995 *J. Phys. A: Math. Gen.* **28** L239
[8] Gómez C and Sierra G 1993 *J. Math. Phys.* **34** 2119
[9] Lyakhovsky V and Mudrov A 1992 *J. Phys. A: Math. Gen.* **25** L1139
[10] Ballesteros A, Herranz F J, Pereña C M, del Olmo M A and Santander M 1995 *J. Phys. A: Math. Gen.* **28** 7113
[11] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1155
[12] Hussin V, Lauzon A and Rideau G 1996 *J. Phys. A: Math. Gen.* **29** 4105
[13] Reshetikhin N Y, Takhtadzhyan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193
[14] Ballesteros A, Celeghini E, Giachetti R, Sorace E and Tarlini M 1993 *J. Phys. A: Math. Gen.* **26** 7495
[15] Hussin V, Lauzon A and Rideau G 1994 *Lett. Math. Phys.* **31** 159
[16] Tjin T 1992 *Int. J. Mod. Phys. A* **7** 6175
[17] Ohn C 1992 *Lett. Math. Phys.* **25** 85
[18] Ballesteros A, Herranz F J, del Olmo M A, and Santander M 1995 *J. Phys. A: Math. Gen.* **28** 941
[19] Ballesteros A, Gromov N A, Herranz F J, del Olmo M A, and Santander M 1995 *J. Math. Phys.* **36** 5916